Characterization for Lipschitz Spaces via commutators of Some Maximal Functions on p-adic Orlicz Spaces

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Abstract: In this paper, the main aim is to demonstrate the boundedness for commutators of fractional maximal function and sharp maximal function in the context of the p-adic version of Orlicz spaces, where the symbols of the commutators belong to the p-adic version of Lipschitz space, whereby some new characterizations for $\Lambda_\beta(\mathbb{Q}_p^n)$ spaces are given.

Keywords: p-adic field; fractional maximal function; sharp maximal function; commutator; Lipschitz space; Orlicz space.

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1 Introduction and main resulats

The research on p-adic fields involves multiple fields, including mathematics, theoretical physics, and computer science. In the field of mathematics, p-adic fields are an important branch of number theory, mainly studying the properties and structures of p-adic numbers [19, 21]. In theoretical physics, the p-adic theory is widely used in quantum mechanics, string theory, and cosmology. Among them, p-adic numbers can describe the states and interactions of particles in quantum mechanics [2, 5, 8], while in string theory they can describe the vibration patterns and interactions of strings. Additionally, p-adic numbers are also used to describe the geometric structure and evolution of the universe in

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cosmology. In computer science, the p-adic theory is applied in cryptography, computer graphics, and data encryption [3, 4, 18]. Among them, p-adic numbers can provide a method of encryption and decryption in cryptography, while in computer graphics they can be used to generate images with complex geometric shapes. In summary, the research on p-adic fields involves multiple fields with a wide range of applications and profound research values and significance.

Assume that $\mathbb{Z}$, $\mathbb{Q}$ be the field of integers and rational numbers, respectively. For a fixed prime number p, the p-adic field $\mathbb{Q}_p$ were originally given by K.Hensel in 1908, is composed of $\mathbb{Q}$ with respect to non-Archimedean p-adic absolute value: let $x = p^\gamma \frac{a}{b}$, where $x \in \mathbb{Q}$ and $\gamma \in \mathbb{Z}$, a and b are non-zero integers which are not divisible by p, the p-adic absolute value is $|x|_p = p^{-\gamma}$.

It is well known that the non-Archimedean p-adic absolute value has many properties similar to the Archimedean absolute value, for instance, positive definiteness, product properties and non-Archimedean p-adic absolute value inequality. Exactly speaking, these properties are shown as follows.

(1) $|x|_p \geq 0$. Specially, $|x|_p = 0$ if and only if $x = 0$;

(2) $|xy|_p = |x|_p|y|_p$;

(3) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. If $|x|_p \neq |y|_p$. then the equality holds and the converse is also true.

Combining Properties (1) and (3), we also obtain the same triangle inequality as Archimedean absolute value, namely, $|x + y|_p \leq |x|_p + |y|_p$.

From the standard p-adic analysis, Nonzero p-adic number x can be written as.

$$x = p^\gamma(a_0 + a_1 p + a_2 p^2 + \cdots) = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad a_j = 0, \cdots, p - 1,$$

where $|x|_p = p^{-\gamma}$ when $a_\gamma \neq 0$. Naturally, the above p-adic number x converges.

In the next time, we need to further consider the n-dimensional p-adic linear space $\mathbb{Q}_p^n$, when $n = 1$, this case is shown in the description above. For any n-dimensional vector $x = (x_1, x_2, \cdot, \cdot, \cdot, x_n)$, where $x_i \in \mathbb{Q}_p$ $(i = 1, ..., n)$, then the following p-adic absolute value is given by

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p.$$

Finally, the p-adic ball is denoted by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\},$$

where the center of p-adic ball $a \in \mathbb{Q}_p^n$ and radius p^γ with $\gamma \in \mathbb{Z}$. The p-adic corresponding sphere is denoted by

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a),$$

specially, if $a, \gamma = 0$, then the $B_0(0)$ and $S_0(0)$ are called the p-adic unit ball and p-adic unit sphere, respectively.

Moreover, when $a = 0$, we usually omit the center of p-adic ball and sphere. On the other hand, from the definition of p-adic ball and sphere, we observe a relation between them, exactly speaking, $B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a)$ and $\mathbb{Q}_p^n \setminus \{0\} = \bigcup_{\gamma \in \mathbb{Z}} S_\gamma$. For any $a_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, it is not difficult to obtain the following equalities

$$a_0 + B_\gamma = B_\gamma(a_0) \text{ and } a_0 + S_\gamma = S_\gamma(a_0) = B_\gamma(a_0) \setminus B_{\gamma-1}(a_0).$$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists Haar measure on $\mathbb{Q}_p^n$, it is easy to know that unique Haar measure $\mathrm{d}x$ on $\mathbb{Q}_p^n$ (up to positive constant multiple) satisfies translation invariant (i.e., $\mathrm{d}(x + a) = \mathrm{d}x$). Here, it means we integrate on p-adic unit ball firstly, such that

$$\int_{B_0} dx = |B_0|_h = 1,$$

where $|B_0|_h$ is denoted by the Haar measure of p-adic unit ball. Generally speaking, for any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, we have

$$\int_{B_\gamma(a)} dx = |B_\gamma(a)|_h = p^{n\gamma}$$

and

$$\int_{S_\gamma(a)} dx = |S_\gamma(a)|_h = p^{n\gamma}(1 - p^{-n}) = |B_\gamma(a)|_h - |B_{\gamma-1}(a)|_h.$$

For more details about the p-adic analysis, we refer readers to [23, 24] and references therein.

As we all known, the study of commutator has caught a lot of attention due to many applications in partial differential equations and hormonic analysis,

Let T be the classic singular integral operator, the Coifman-Rochberg-Weiss type commutator $[b, T]$ generated by T and a suitable function b is defined by

$$[b, T]f = bT(f) - T(bf).$$

It is well-known that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded on $L^s(\mathbb{R}^n)$ for $1 < s < \infty$, that's because Coifman, Rochberg and Weiss [9](see also [16]). In [16], Janson proved that the necessary and sufficient condition for the boundedness of commutators from $L^s(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ ($1 < s < \frac{n}{\beta}$ and $\frac{1}{s} - \frac{1}{q} = \frac{\beta}{n}$ with $0 < \beta < 1$) is $b \in \Lambda_\beta(\mathbb{R}^n)$, thus giving some characterizations of Lipschitz space $\Lambda_\beta(\mathbb{R}^n)$.

Let $0 < \alpha < n$, we define the p-adic fractional maximal operator as

$$M_\alpha^p(f)(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}} \int_{B_\gamma(x)} |f(y)| dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$.

Then the maximal commutator of M_α^p with b is given by

$$M_{\alpha,b}^p(f)(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}} \int_{B_\gamma(x)} |b(x) - b(y)||f(y)| dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$.

Furthermore, assume that $b : \mathbb{Q}_p^n \to \mathbb{R}$ and $f : \mathbb{Q}_p^n \to \mathbb{R}$ are measurable mappings, then the nonlinear commutators of fractional maximal operator can be defined as follows:

$$[b, M_\alpha^p]f(x) = b(x)M_\alpha^p(f)(x) - M_\alpha^p(bf)(x).$$

If $\alpha = 0$, we have $[b, M^p] = [b, M_0^p]$ and $M_b^p = M_{0,b}^p$.

It is easy to see that $M_{\alpha,b}^p$ and $[b, M_\alpha^p]$ essentially differ from each other. For instant, $M_{\alpha,b}^p$ is positive and sublinear, however, $[b, M_\alpha^p]$ is neither positive nor sublinear.

The fractional maximal operator and its commutators generated by functions of different forms have been considered by many authors and applied to various aspects, for example, [1, 6, 10–13, 31]. Exactly speaking, for the case of symbol $b \in \mathrm{BMO}(\mathbb{R}^n)$, Bastero et al. [6] obtained the boundedness of $[b, M]$ in $L^q(\mathbb{R}^n)$ for $1 < q < \infty$. Zhang and Wu further in [28–30]considered the same problem for the commutator of fractional maximal function and obtained the mentioned results on variable Lebesgue spaces, as well as the commutator $[b, M^\sharp]$. When the symbol $b \in \Lambda_\beta(\mathbb{R}^n)$. Guliyev, Deringoz and Hasanov [12] gave the boundedness of fractional maximal commutator and nonlinear commutator of fractional maximal function on Orlicz space. In [31] by Zhang et al further extended some results of [12] to nonnegative Lipschitz function and obtained the characterization for the boundedness of the maximal commutator $M_{\alpha,b}$.

Actually, the commutators of fractional maximal operator $[b, M_\alpha^p]$ have been studied by many authors in the p-adic linear space, for instance, [7, 14, 15, 25, 26] etc. Recently,

He and Li [15] gave the necessary and sufficient conditions for boundedness of some commutators of maximal function on the p-adic linear spaces, besides, Wu and Chang [26, 27] not only extended the results to commutators of the fractional maximal function, but also the results were extended to p-adic variable Lebesgue spaces, as well as the commutator $[b, M^\sharp]$. However, the study of Orlicz spaces in p-adic linear spaces is quite a few, which look worthy of further investigations.

Encouraged by the above-mentioned literature, The purpose of this article is to obtain the boundedness for commutators of fractional maximal function and sharp maximal function in the context of the p-adic version of Orlicz spaces, where the symbols of the commutators belong to the p-adic version of Lipschitz space, whereby some new characterizations for $\Lambda_\beta(\mathbb{Q}_p^n)$ spaces are given.

our results are stated as follows:

The following result, futhermore, let $\alpha > 0$, for a fixed p-adic ball B_*, the fractional maximal function with respect to B_* of locally integrable function f is given by

$$M_{\alpha,B_*}^p(f)(x) = \sup_{\substack{\gamma \in \mathbb{Z} \\ B_\gamma(x) \subset B_*}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}} \int_{B_\gamma(x)} |f(y)| dy,$$

where the supremum is taken over all the p-adic ball $B_\gamma(x)$ with $B_\gamma(x) \subset B_*$ for a fixed p-adic ball B_*. If $\alpha = 0$, $M_{B_*}^p = M_{0,B_*}^p$.

Naturally, we need to consider the follwing result of boundedness of the nonlinear commutators $[b, M_\alpha^p]$ and $M_{\alpha,b}^p$

Theorem 1.1 Assume that $b \in L_{loc}^1(\mathbb{Q}_p^n)$, $0 \le \alpha < n, 0 < \beta < 1$, $0 < \alpha + \beta < n$, suppose that Φ and Ψ are Young functions satisfying $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$, then the following statements are equivalent:

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$.
(2) $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.
(3) There exists a constant $C > 0$ such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - M_{B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \le C. \tag{1.1}$$

(4) There exists a constant $C > 0$ such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}} \int_{B_\gamma(x)} |b(y) - M_{B_\gamma(x)}^p(b)(y)| dy \le C \tag{1.2}$$

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Remark 1 Theorem 1.1 obtains new characterizations of the non-negative Lipschitz functions. Some similar conclusions were shown in the Lebesgue space and variable exponent Lebesgue space in [15, 25]. Let $\Psi^{-1}(|B_\gamma(x)|_h^{-1}) = \|\chi_{B_\gamma(x)}\|_{L^\Psi(\mathbb{Q}_p^n)}^{-1}$ (see Lemma 2.3 below), then (1.1) are equivalent to the following inequality.

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} |B_\gamma(x)|_h^{-\frac{\beta}{n}} \frac{\|(b - M_{B_\gamma(x)}^p(b))\chi_{B_\gamma(x)}\|_{L^\Psi(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^\Psi(\mathbb{Q}_p^n)}} \le C,$$

which can be compared with some results we know. For instant, let $q(\cdot)$ be a variable exponent and satisfy the Hardy-Littlewood maximal operator is bounded on $L^{q(\cdot)}(\mathbb{Q}_p^n)$, then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$. if and only if

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} |B_\gamma(x)|_h^{-\frac{\beta}{n}} \frac{\|(b - M_{B_\gamma(x)}^p(b))\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \le C,$$

For the case of $\Phi(t) = t^r, \Psi(t) = t^q$. the following result can be obtained. We also see [25]

Corollary 1.1 Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n), 0 \le \alpha < n$. If $1 < r < \frac{n}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{r} - \frac{\alpha+\beta}{n}$, then the following statements are equivalent.

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$.
(2) $[b, M_\alpha^p]$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$.
(3) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - M_{B_\gamma(x)}^p(b)(y)|^q dy \right)^{\frac{1}{q}} \le C.$$

(4) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}} \int_{B_\gamma(x)} |b(y) - M_{B_\gamma(x)}^p(b)(y)| dy \le C$$

For the case of $\alpha = 0$ at theorem 1.2. we have the following result which extends the result of Theorem 4 in [15] to the Orlicz space.

Corollary 1.2 Assume that $b \in L^1_{loc}(\mathbb{Q}_p^n)$, suppose that Φ and Ψ are Young functions satisfying $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma\beta}\Phi^{-1}(p^{-\gamma n})$, then the following statements are equivalent.

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(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$.

(2) $[b, M^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(3) There exists a constant $C > 0$ such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - M_{B_\gamma(x)}^p(b)(\cdot)\|_{L^\Phi(B_\gamma(x))} \leq C$$

(4) There exists a constant $C > 0$ such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - M_{B_\gamma(x)}^p(b)(y)|dy \leq C$$

Theorem 1.2 Assume that $b \in L_{loc}^1(\mathbb{Q}_p^n)$, $0 \leq \alpha < n, 0 < \beta < 1$, $0 < \alpha + \beta < n$, suppose that Φ and Ψ are Young functions satisfying $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$, then the following statements are equivalent.

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

(2) $M_{\alpha,b}^p$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(3) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - b_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} \leq C. \tag{1.3}$$

(4) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}} \int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}|dy \leq C. \tag{1.4}$$

Remark 2 Theorem 1.3 obtains new characterizations of Lipschitz functions. Some similar results can be found in p-adic variable exponent Lebesgue spaces in [26].

For the case of $\alpha = 0$ at theorem 1.3. we have the following result, which extends the result of Theorem 1 in [15] to Orlicz space.

Corollary 1.3 Assume that $b \in L_{loc}^1(\mathbb{Q}_p^n)$, $0 < \beta < 1$, suppose that Φ and Ψ are Young functions satisfying $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma\beta}\Phi^{-1}(p^{-\gamma n})$, then the following statements are equivalent.

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

(2) M_b^p is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(3) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - b_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} \leq C.$$

(4) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}} \int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}| dy \leq C.$$

In order to introduce the following results, we also need to give the p-adic version of sharp maximal function $M_p^\sharp$, for a locally integrable function f on $\mathbb{Q}_p^n$, then, in[17], define that

$$M_p^\sharp(f)(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}| dy.$$

Where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $f_{B_\gamma(x)} = \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} f(y) dy$. The following theorem introduce the commutator of sharp maximal function and Lipschitz function b in Orlicz space.

Theorem 1.3 Assume that $b \in L^1_{loc}(\mathbb{Q}_p^n)$, suppose that Φ are Young functions satisfying $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma \beta} \Phi^{-1}(p^{-\gamma n})$, then the following statements are equivalent.

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$.

(2) $[b, M_p^\sharp]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(3) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(\cdot)\|_{L^\Phi(B_\gamma(x))} \leq C. \qquad (1.5)$$

(4) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)| dy \leq C. \qquad (1.6)$$

Remark 3 Theorem 1.4 obtains new characterizations of the non-negative Lipschitz functions that differ from Theorem 1.2. Some similar conclusions were shown in the Lebesgue space in [14, 27].

For the case of $\Phi(t) = t^r$. the following result can be obtained. We also see [27]

Corollary 1.4 Assume that $b \in L^1_{loc}(\mathbb{Q}_p^n)$, If $1 < r < \frac{n}{\beta}$ and $\frac{1}{q} = \frac{1}{r} - \frac{\beta}{n}$, then the following statements are equivalent:

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$

(2) $[b, M_p^\sharp]$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$.

(3) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \Big(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|^q dy \Big)^{\frac{1}{q}} \leq C.$$

(4) There exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)| dy \leq C$$

Throughout this paper, the letter C always takes place of a constant independent of the primary parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter $|E|_h$ will always denote the Haar measure of a measurable set E on $\mathbb{Q}_p^n$ and by χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{Q}_p^n$.

2 Preliminaries

2.1 p-adic function spaces

Assume that $1 \leq q < \infty$, we denote $L^q(\mathbb{Q}_p^n)$ as the p-adic Lebesgue space, the space of all functions f is in the locally L^q space with finite norm

$$\|f\|_{L^q(\mathbb{Q}_p^n)} = \Big(\int_{\mathbb{Q}_p^n} |f(x)|^q dx \Big)^{\frac{1}{q}}.$$

In addition, for $q = \infty$ and denote $L^\infty(\mathbb{Q}_p^n)$ as the set of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \operatorname*{ess\,sup}_{x \in \mathbb{Q}_p^n} |f(x)| = \inf\{\lambda > 0 : |x \in \mathbb{Q}_p^n : |f(x)| > \lambda|_h\} < \infty.$$

Here, if the limit exists, the integral in above equation is defined as follows:

$$\int_{\mathbb{Q}_p^n} |f(x)|^q dx = \lim_{\gamma\to\infty} \int_{B_\gamma(0)} |f(x)|^q dx = \lim_{\gamma\to\infty} \sum_{-\infty < k \le \gamma} \int_{S_k(0)} |f(x)|^q dx.$$

In particular, since $\mathbb{Q}_p^n = \bigcup_{\gamma=-\infty}^{+\infty} S_\gamma$, and $d(tx) = |t|_p^n dx (t \in \mathbb{Q}_p \setminus \{0\})$, if $f \in L^1(\mathbb{Q}_p^n)$, then

$$\int_{\mathbb{Q}_p^n} f(x)dx = \sum_{\gamma=-\infty}^{+\infty} \int_{S_\gamma} f(x)dx$$

and

$$\int_{\mathbb{Q}_p^n} f(tx)dx = \frac{1}{|t|_p^n} \int_{\mathbb{Q}_p^n} f(x)dx.$$

If a left-continuous , convex and increasing function $\Phi : [0,\infty) \to [0,\infty]$ satisfies $\lim_{t\to 0+} \Phi(t) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$, then we shall call $\Phi(t)$ Young function.

Definition 2.1 Let Φ be a Young's function.
(1) Denote by $\mathcal{Y}$ the set of all functions $\Phi : [0,\infty) \to [0,\infty]$ such that

$$0 < \Phi(t) < \infty, \quad 0 < t < \infty.$$

(2) Denote by ∇_2 the set of all functions $\Phi : [0,\infty) \to [0,\infty]$ such that for some $K > 1$

$$\Phi(t) \le \frac{1}{2K}\Phi(Kt), \quad t \ge 0.$$

If $\Phi \in \mathcal{Y}$, it is easy to show that Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to $[0,\infty)$, then there exists inverse function $\Phi^{-1}(s) = \inf\{t \ge 0 : \Phi(t) > s\}$, $s \in [0,\infty]$.

Definition 2.2 Given a Young's function Φ, the Young's complement $\bar{\Phi}$ is defined for $0 \le x < \infty$ by

$$\bar{\Phi}(x) = \begin{cases} \sup_{0 \le y < \infty}(xy - \Phi(y)) & \text{if } y \in [0,\infty], \\ \infty & \text{if } y = \infty. \end{cases}$$

The definition and properties of p-adic Orlicz space are mentioned below.

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We define the Orlicz space by $L^{\Phi}(\mathbb{Q}_p^n)$ to be the set of all measurable function $g :$ $\mathbb{Q}_p^n \to \mathbb{R}$ such that for some $\beta > 0$

$$\int_{\mathbb{Q}_p^n} \Phi(\frac{|g(x)|}{\beta})dx < \infty.$$

When Φ is a Young function, which is a convex Orlicz function, the property

$$\|f\|_{\Phi} = \inf\{\alpha > 0 : \int_{\mathbb{Q}_p^n} \Phi(\frac{f(x)}{\alpha})dx \leq 1\}$$

is well known as the Luxemburg norm that can be found in [22].

If $\Phi(t) = t^q$, $1 \leq q < \infty$, then $L^{\Phi}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$. If $\Phi(t) = 0$, $0 \leq t \leq 1$ and $\Phi(t) = \infty$, $t > 1$, then $L^{\Phi}(\mathbb{Q}_p^n) = L^{\infty}(\mathbb{Q}_p^n)$.

2.2 Auxiliary propositions and lemmas

In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

The following Lemma gives the basic definition of p-adic Lipschitz spaces, and the third part can be found in [15].

Lemma 2.1 Assume $0 < \beta < 1$ and $\mathbb{Q}_p^n$ be an n-dimensional p-adic linear space.
(1) The p-adic version of homogeneous Lipschitz spaces $\Lambda_{\beta}(\mathbb{Q}_p^n)$ is defined by

$$\Lambda_{\beta}(\mathbb{Q}_p^n) := \{f \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{Q}_p^n) : \|f\|_{\Lambda_{\beta}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\Lambda_{\beta}(\mathbb{Q}_p^n)} = \sup_{x,y \in \mathbb{Q}_p^n, \ x \neq y} \frac{|f(x) - f(y)|}{|x - y|_p^{\beta}}.$$

(2) If $1 \leq q < \infty$, the p-adic version of Lipschitz spaces $\mathrm{Lip}_{\beta}^q(\mathbb{Q}_p^n)$ is defined by

$$\mathrm{Lip}_{\beta}^q(\mathbb{Q}_p^n) := \{f \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{Q}_p^n) : \|f\|_{\mathrm{Lip}_{\beta}^q(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}_{\beta}^q(\mathbb{Q}_p^n)} = \sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_{\gamma}(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_{\gamma}(x)|_h} \int_{B_{\gamma}(x)} |f(y) - f_{B_{\gamma}(x)}|^q dy\right)^{\frac{1}{q}}$$

.

(3) The homogeneous Lipschitz space $\Lambda_{\beta}(\mathbb{Q}_p^n)$ are equivalent to the above $\mathrm{Lip}_{\beta}^q(\mathbb{Q}_p^n)$, that's to say,

$$\|f\|_{\Lambda_{\beta}(\mathbb{Q}_p^n)} \approx \|f\|_{\mathrm{Lip}_{\beta}^q(\mathbb{Q}_p^n)}.$$

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The following result in the Euclidean space can be found in [22], by the similar way we can obtain the version of p-adic Hölder's inequality on Orlicz space.

Lemma 2.2 Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic linear space. For a Young function Φ with its complementary function Ψ. Assume that measurable function $f \in L^{\Phi}(\mathbb{Q}_p^n)$ and $g \in L^{\Psi}(\mathbb{Q}_p^n)$. Then there exists a positive constant C such that

$$\int_{\mathbb{Q}_p^n} |f(x)g(x)|dx \leq C\|f\|_{L^{\Phi}(\mathbb{Q}_p^n)}\|g\|_{L^{\Psi}(\mathbb{Q}_p^n)}.$$

By elementary calculations we obtain the following results for the characteristic function $\chi_{B_{\gamma}(x)}$.

Lemma 2.3 Assume that Φ be a Young function and $B_{\gamma}(x)$ be a set in $\mathbb{Q}_p^n$ with finite Haar measure. Then

$$\|\chi_{B_{\gamma}(x)}\|_{L^{\Phi}(\mathbb{Q}_p^n)} = \frac{1}{\Phi^{-1}(|B_{\gamma}(x)|_h^{-1})}.$$

The following results can be obtained by Lemmas 2.2, 2.3, thus we omit the proof.

Lemma 2.4 For a p-adic ball $B_{\gamma}(x)$ and a Young function Φ. The following inequality is hold:

$$\int_{B_{\gamma}(x)} |f(y)|dy \leq C|B_{\gamma}(x)|_h \Phi^{-1}(|B_{\gamma}(x)|_h^{-1})\|f\|_{L^{\Phi}(B_{\gamma}(x))}$$

The following results can be found in [20] in the Euclidean spaces, by the similar way we can obtain the following Lemma and omit the proof.

Lemma 2.5 Assume that $0 < \alpha < n, \Phi, \Psi$ be Young functions and $\Phi \in \mathcal{Y} \cap \nabla_2$. Then for all $\gamma \in \mathbb{Z}$, there exist positive constant C does not depend on γ, such that the fractional maximal operator M_{α}^p is bounded from $L^{\Phi}(\mathbb{Q}_p^n)$ to $L^{\Psi}(\mathbb{Q}_p^n)$ if and only if the condition $p^{\gamma\alpha}\Phi^{-1}(p^{-\gamma n}) \leq C\Psi^{-1}(p^{-\gamma n})$.

The following Lemma (we can see [26],Lemma 2.11) can be obtained.

Lemma 2.6 Let $b \in L^1_{loc}(\mathbb{Q}_p^n)$. For any fixed p-adic ball $B_{\gamma}(x) \subset \mathbb{Q}_p^n$.
(1) If $0 \leq \alpha < n$, then for all $y \in B_{\gamma}(x)$, we have

$$M_{\alpha}^p(b\chi_{B_{\gamma}(x)})(y) = M_{\alpha, B_{\gamma}(x)}^p(b)(y)$$

and

$$M_{\alpha}^p(\chi_{B_{\gamma}(x)})(y) = M_{\alpha, B_{\gamma}(x)}^p(\chi_{B_{\gamma}(x)})(y) = |B_{\gamma}(x)|_h^{\frac{\alpha}{n}}$$

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(2) Then for any $y \in B_\gamma(x)$, we have

$$|b_{B_\gamma(x)}| \leq |B_\gamma(x)|_h^{-\frac{\alpha}{n}} M_{\alpha, B_\gamma(x)}(b)(y)$$

(3) Let $E = \{y \in B_\gamma(x) : b(y) \leq b_{B_\gamma(x)}\}$ and $F = B_\gamma(x) \setminus E = \{y \in B_\gamma(x) : b(y) > b_{B_\gamma(x)}\}$. Then the following equality is trivially true

$$\int_E |b(y) - b_{B_\gamma(x)}|dy = \int_F |b(y) - b_{B_\gamma(x)}|dy$$

Finally, the following results plays a key role in proving our main theorems. For some details, see Lemma 2.7 in [26].

Lemma 2.7 Assume $0 < \beta < 1$ and b be a locally integrable function on $\mathbb{Q}_p^n$. Then the following assertions are equivalent:

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$.
(2) For all $1 \leq s < \infty$, there exists a positive constant C such that

$$\sup_{\substack{\gamma \in \mathbb{Z} \\ x \in \mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - M_{B_\gamma(x)}^p(b)(y)|^s dy \right)^{\frac{1}{s}} \leq C \qquad (2.1)$$

(3) (2.1) holds for some $1 \leq s < \infty$.

Lemma 2.8 Let $0 \leq \alpha < n$, $0 < \beta < 1$, $0 < \alpha + \beta < n$ and f be a locally integrable function on $\mathbb{Q}_p^n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, then, for any $x \in \mathbb{Q}_p^n$ such that $M_\alpha^p(f)(x) < \infty$, we have

$$|[b, M_\alpha^p](f)(x)| \leq \|b\|_{\Lambda_\beta} M_{\alpha+\beta}^p(f)(x).$$

In order to obtain our theorems, we need to prove the following result.

Lemma 2.9 Assume that $b \in L_{loc}^1(\mathbb{Q}_p^n)$, $0 \leq \alpha < n$, $0 < \beta < 1$, $0 < \alpha + \beta < n$, suppose that Φ and Ψ are Young functions satisfying $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$, then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ is necessary for the boundedness of $[b, M_\alpha^p]$ from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

Proof: For any fixed p-adic ball $B_\gamma(x)$ and any $y \in B_\gamma(x)$, by applying Lemma 2.6, we have

$$M_\alpha^p(b\chi_{B_\gamma(x)})(y) = M_{\alpha, B_\gamma(x)}^p(b)(y) \text{ and } M_\alpha^p(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\frac{\alpha}{n}}.$$

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Since $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, by Lemma 2.3, then,

$$|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$= |B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})$$

$$\|b(\cdot)M_\alpha^p(\chi_{B_\gamma(x)})(\cdot) - M_\alpha^p(b\chi_{B_\gamma(x)})(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$= |B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|[b, M_\alpha^p](\chi_{B_\gamma(x)})\|_{L^\Psi(B_\gamma(x))}$$

$$\leq C|B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|\chi_{B_\gamma(x)}\|_{L^\Phi(B_\gamma(x))}$$

$$\leq C.$$

In view of the proof of Lemma 3.1 in [26], we have

$$\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\int_{B_\gamma(x)}|b(y) - b_{B_\gamma(x)}|dy \leq \frac{2}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\int_{B_\gamma(x)}|b(y)-$$

$$|B_\gamma(x)|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\gamma(x)}^p(b)(y)|dy.$$

Thus, by applying Lemma 2.2, we get

$$\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\int_{B_\gamma(x)}|b(y) - b_{B_\gamma(x)}|dy$$

$$\leq C|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$\leq C.$$

It follows from Lemma 2.1 that $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

3 Proof of the principal results

Proof of Theorem 1.1 *Since the statements* $(1) \Longleftrightarrow (4)$ *directly are obtained by Lemma 2.7, we only need to prove* $(1) \Longrightarrow (2)$, $(2) \Longrightarrow (3)$ *and* $(3) \Longrightarrow (4)$.

$(1) \Longrightarrow (2)$, *since* $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, *and* $b \geq 0$, *by using Lemma 2.8, for almost every* $x \in \mathbb{Q}_p^n$ *and for all functions from* $f \in L_{\text{loc}}^1(\mathbb{Q}_p^n)$

$$|[b, M_\alpha^p](f)(x)| \leq \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}M_{\alpha+\beta}^p(f)(x).$$

It follows from Lemma 2.5 that $[b, M_\alpha^p]$ *is bounded from* $L^\Phi(\mathbb{Q}_p^n)$ *to* $L^\Psi(\mathbb{Q}_p^n)$.

$(2) \Longrightarrow (3)$, *We divide the proof into two cases according to the range of* α.

Case 1. If $\alpha = 0$. For any fixed p-adic ball $B_\gamma(x)$ and any $y \in B_\gamma(x)$, we claim that (see also [15])

$$M^p(\chi_{B_\gamma(x)})(y) = \chi_{B_\gamma(x)}(y), \quad M^p(b\chi_{B_\gamma(x)})(y) = M^p_{B_\gamma(x)}(b)(y). \qquad (3.1)$$

Then

$$b(y) - M^p_{B_\gamma}(b)(y) = b(y)M^p(\chi_{B_\gamma(x)})(y) - M^p(b\chi_{B_\gamma(x)})(y) = [b, M^p](\chi_{B_\gamma(x)})(y).$$

By using the condition $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n}) = p^{\beta\gamma}\Phi^{-1}(p^{-\gamma n})$ and Lemma 2.3, we get

$$|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - M^p_{B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$= |B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|[b, M^p](\chi_{B_\gamma(x)})(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$\leq C|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|(\chi_{B_\gamma(x)})\|_{L^\Phi(B_\gamma(x))}$$

$$\leq C.$$

Thus, we obtain (1.1).

Case 2. If $0 < \alpha < n$. For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$,

$$|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - M^p_{B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$\leq |B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}}M^p_{\alpha,B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$+ |B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|M^p_{B_\gamma(x)}(b)(\cdot) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}}M^p_{\alpha,B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$:= I_1 + I_2.$$

We first consider I_1. For any $y \in B_\gamma(x)$, it follows from Lemma 2.6 that

$$M^p_\alpha(b\chi_{B_\gamma(x)})(y) = M^p_{\alpha,B_\gamma(x)}(b)(y) \text{ and } M^p_\alpha(\chi_{B_\gamma(x)})(y) = M^p_{\alpha,B_\gamma(x)}(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\frac{\alpha}{n}}.$$

Then, for any $y \in B_\gamma(x)$,

$$b(y) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}}M^p_{\alpha,B_\gamma(x)}(b)(y) = |B_\gamma(x)|_h^{-\frac{\alpha}{n}}(b(y)|B_\gamma(x)|_h^{\frac{\alpha}{n}} - M^p_{\alpha,B_\gamma(x)}(b)(y))$$

$$= |B_\gamma(x)|_h^{-\frac{\alpha}{n}}(b(y)M^p_\alpha(\chi_{B_\gamma(x)})(y) - M^p_\alpha(b\chi_{B_\gamma(x)})(y))$$

$$= |B_\gamma(x)|_h^{-\frac{\alpha}{n}}[b, M^p_\alpha](\chi_{B_\gamma(x)})(y).$$

In view of $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, by using Lemma 2.3 and the hypothesis $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$, then

$$
\begin{aligned}
I_1 &= |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&= |B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|[b, M_\alpha^p](\chi_{B_\gamma(x)})\|_{L^\Psi(B_\gamma(x))} \\
&\leq C|B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|\chi_{B_\gamma(x)}\|_{L^\Phi(B_\gamma(x))} \\
&\leq C.
\end{aligned}
$$

Next, we further estimate I_2. For any $y \in B_\gamma(x)$, by using (3.1), we have

$$
\begin{aligned}
&|M_{B_\gamma(x)}^p(b)(y) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\gamma(x)}^p(b)(y)| \\
&= |B_\gamma(x)|_h^{-\frac{\alpha}{n}} |M_{\alpha,B_\gamma(x)}^p(b)(y) - |B_\gamma(x)|_h^{\frac{\alpha}{n}} M_{B_\gamma(x)}^p(b)(y)| \\
&= |B_\gamma(x)|_h^{-\frac{\alpha}{n}} |M_\alpha^p(b\chi_{B_\gamma(x)})(y) - M_\alpha^p(\chi_{B_\gamma(x)})(y) M^p(b\chi_{B_\gamma(x)})(y)| \\
&\leq |B_\gamma(x)|_h^{-\frac{\alpha}{n}} |M_\alpha^p(b\chi_{B_\gamma(x)})(y) - |b(y)| M_\alpha^p(\chi_{B_\gamma(x)})(y)| \\
&\quad + |B_\gamma(x)|_h^{-\frac{\alpha}{n}} ||b(y)| M_\alpha^p(\chi_{B_\gamma(x)})(y) - M_\alpha^p(\chi_{B_\gamma(x)})(y) M^p(b\chi_{B_\gamma(x)})(y)| \\
&= |B_\gamma(x)|_h^{-\frac{\alpha}{n}} |M_\alpha^p(|b(y)|\chi_{B_\gamma(x)})(y) - |b(y)| M_\alpha^p(\chi_{B_\gamma(x)})(y)| \\
&\quad + |B_\gamma(x)|_h^{-\frac{\alpha}{n}} M_\alpha^p(\chi_{B_\gamma(x)})(y) ||b(y)| M^p(\chi_{B_\gamma(x)})(y) - M^p(b\chi_{B_\gamma(x)})(y)| \\
&= |B_\gamma(x)|_h^{-\frac{\alpha}{n}} |[|b|, M_\alpha^p]\chi_{B_\gamma(x)}(y)| + |[|b|, M^p]\chi_{B_\gamma(x)}(y)|.
\end{aligned}
$$

Now, since $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, then it follows from Lemma 2.9 that $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, which implies $|b| \in \Lambda_\beta(\mathbb{Q}_p^n)$.

By using Lemma 2.6 and Lemma 2.8, for any $y \in B_\gamma(x)$, we have

$$
|[|b|, M_\alpha^p]\chi_{B_\gamma(x)}(y)| \leq \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta}^p(\chi_{B_\gamma(x)})(y) \leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} |B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}
$$

and

$$
|[|b|, M^p]\chi_{B_\gamma(x)}(y)| \leq \|b\|_{\Lambda_\beta} M_\beta^p(\chi_{B_\gamma(x)})(y) \leq C\|b\|_{\Lambda_\beta} |B_\gamma(x)|_h^{\frac{\beta}{n}}.
$$

By using Lemma 2.3, we obtain

$$
\begin{aligned}
I_2 &= |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|M_{B_\gamma(x)}^p(b)(\cdot) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&\leq C(\Psi^{-1}(|B_\gamma(x)|_h^{-1}) |B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n}} \|[|b|, M_\alpha^p]\chi_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} \\
&\quad + |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|[|b|, M^p]\chi_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))})
\end{aligned}
$$

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$$\leq C(\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}(\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|\chi_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))}))$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}.$$

By the above I_1, I_2 estimates, which leads us to (1.1) since $B_\gamma(x)$ is arbitrary.

(3) $\Longrightarrow$ (4). In order to deduce (1.2) from (1.1). Assume (1.1) holds, for any fixed p-adic ball $B_\gamma(x)$, by using Lemma 2.2 and (1.1), we obtain

$$\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}}\int_{B_\gamma(x)}|b(y)-M_{B_\gamma(x)}^p(b)(y)|dy$$
$$\leq C|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h)\|b(\cdot)-M_{B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}$$
$$\leq C.$$

The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2 *Since the implications* (1) $\Longleftrightarrow$ (4) *follow readily from Corollary 1.3 in [26], and* (3) $\Longleftrightarrow$ (4) *follows from Lemma 2.2 directly, we only need to prove* (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3).

(1) $\Longrightarrow$ (2), *since* $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, *for any p-adic ball* $B_\gamma(x) \subset \mathbb{Q}_p^n$,

$$M_{\alpha,b}^p(f)(x) = \sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}}\int_{B_\gamma(x)}|b(x)-b(y)||f(y)|dy$$
$$\leq \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}\sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}}\int_{B_\gamma(x)}|x-y|_p^\beta|f(y)|dy$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}\sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha+\beta}{n}}}\int_{B_\gamma(x)}|f(y)|dy$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}M_{\alpha+\beta}^p(f)(x).$$

Thus by Lemma 2.5 we get that $M_{\alpha,b}^p$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(2) $\Longrightarrow$ (3). *For any fixed p-adic ball* $B_\gamma(x)$, *we have for every* $y \in B_\gamma(x)$,

$$|b(y)-b_{B_\gamma(x)}| \leq \frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}|b(y)-b(z)|dz$$
$$= \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha}{n}}}\frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}}\int_{B_\gamma(x)}|b(y)-b(z)|\chi_{B_\gamma(x)}(z)dz$$
$$\leq \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha}{n}}}M_{\alpha,b}^p(\chi_{B_\gamma(x)})(y).$$

Since $M_{\alpha,b}^p$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, noting that $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$ and by applying Lemma 2.3, we have

$$|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - b_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))}$$

$$\leq \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|M_{\alpha,b}^p(\chi_{B_\gamma(x)})(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$\leq C\frac{1}{|B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|\chi_{B_\gamma(x)}\|_{L^\Phi(B_\gamma(x))}$$

$$\leq C,$$

which implies (1.3). Thus, we finish the proof of Theorem 1.2.

Proof of Theorem 1.3 Since the implications (1) $\Longleftrightarrow$ (4) follow readily from Theorem 1.4 in [14], we only need to prove (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4).

(1) $\Longrightarrow$ (2). Assume $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, and $b \geq 0$, for any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, the following estimate was obtained in [14]:

$$|[b, M_p^\sharp]f(x)| \leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}M_\beta^p(f)(x).$$

Therefore, by using Lemma 2.5, we obtain that $[b, M_p^\sharp]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(2) $\Longrightarrow$ (3). Assume that $[b, M_p^\sharp]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$. In order to prove (1.6). For any fixed p-adic ball $B_\gamma(x)$, we have (see also [14])

$$M_p^\sharp(\chi_{B_\gamma(x)})(y) = \frac{2(p-1)}{p^2}, \quad y \in B_\gamma(x).$$

Then, for all $y \in B_\gamma(x)$,

$$b(y) - \frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})(y) = \frac{p^2}{2(p-1)}(\frac{2(p-1)}{p^2}b(y) - M_p^\sharp(b\chi_{B_\gamma(x)})(y))$$

$$= \frac{p^2}{2(p-1)}(b(y)M_p^\sharp(\chi_{B_\gamma(x)})(y) - M_p^\sharp(b\chi_{B_\gamma(x)})(y))$$

$$= \frac{p^2}{2(p-1)}[b, M_p^\sharp](\chi_{B_\gamma(x)})(y).$$

Since $[b, M_p^\sharp]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, then by applying Lemma 2.3 and noting that $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma\beta}\Phi^{-1}(p^{-\gamma n})$, we have

$$|B_\gamma(x)|_h^{-\frac{\beta}{n}}\Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - \frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$= \frac{p^2}{2(p-1)} |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|[b, M_p^\sharp](\chi_{B_\gamma(x)})\|_{L^\Psi(B_\gamma(x))}$$

$$\leq C |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|\chi_{B_\gamma(x)}\|_{L^\Phi(\mathbb{Q}_p^n)}$$

$$\leq C.$$

Then we obtain (1.5).

(3) $\implies$ (4). Assume (1.5) holds, we need to prove (1.6). For any fixed p-adic ball $B_\gamma(x)$, according to Lemma 2.2 and (1.5), we have

$$\frac{1}{|B_\gamma(x)|_h^{1+\frac{\beta}{n}}} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)| dy$$

$$\leq C |B_\gamma(x)|_h^{-\frac{\beta}{n}} \Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(\cdot)\|_{L^\Psi(B_\gamma(x))}$$

$$\leq C,$$

which implies (1.6) since the constant C is independent of $B_\gamma(x)$. Therefore, we finish the proof of Theorem 1.3.

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Conflict of interest

The authors state that there is no conflict of interest.

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Author contributions

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